JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 23, 365–383 (1968)

A Volterra Integral Equation in the Stability of Some Linear Hereditary Phenomena

NESTOR DISTEFANO

Department of Electrical Engineering, University of Southern California, Los Angeles, California 90007

Submitted by R. Bellman

1. INTRODUCTION

A number of interesting problems in the theory of hereditary systems arisen when the "forcing term" depends not only on the external excitation but also on the deviation of the system from a natural position of equilibrium. In those cases the problem can be generally reduced to the investigation of a system of Volterra integral equations whose kernels generally include functions of the external forces. A simple example, which however preserves the essence of much more complicated systems—thus clarifying the structure of the pertinent equation—comes from the field of hereditary mechanics, upon considering the model in Fig. 1. In this model, the deflection of the

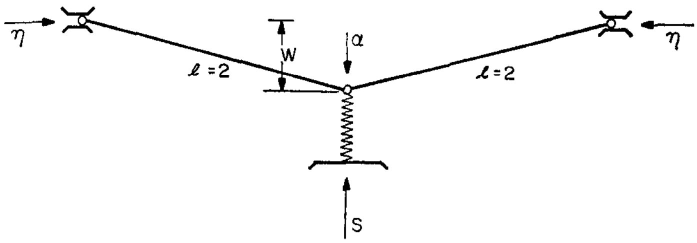

FIG. 1

two hinged bars system, due to the action of the forces α and η, is prevented by a viscoelastic spring which in turn reacts with a force S. It is clear that in general the deflection w will result a nonlinear functional of the axial load η. Our purpose, next, is to write an equation accounting for this dependence, under general assumptions regarding the nature of the viscoelastic spring.

* This work was supported by the National Institutes of Health under Grant No. GM-14633-02.

Disregarding inertia effects, equilibrium requires that at time t, the force S in the spring be given by

$$S(t) = \alpha(t) + \eta(t) \frac{w(t)}{\sqrt{1 - \left(\frac{w}{2}\right)^2}}. \tag{1.1}$$

The relationship force-displacement in the spring is assumed to be given in terms of a functional

$$w(t) = \mathfrak{g}[S_{-\infty}^{t}(\tau)], \tag{1.2}$$

which accounts for the hereditary effects in the system.

Substitution of S given by Eq. (1.1) in Eq. (1.2), furnishes a functional relation between the state of the system, represented by w in this case, and the external excitation α and η.

The linearized version of this problem may be obtained by considering

$$S = \alpha + \eta w \tag{1.3}$$

and

$$w(t) = S(t) + \int_0^t f(t, \tau)\, S(\tau)\, d\tau, \tag{1.4}$$

instead of Eqs. (1.1) and (1.2), respectively. The precise nature of the linear functional approximation appearing in Eq. (1.4) will not be discussed here. We note that the lower limit of integration comes from tacitly assuming the existence of a natural state of equilibrium $w = cte$ when $\alpha(t) = \eta(t) \equiv 0$, $t \leqslant 0$.

Elimination of S between Eqs. (1.3) and (1.4) yields

$$(1 - \eta)\, w - \int_0^t f(t, \tau)\, \eta(\tau)\, w(\tau)\, d\tau = g(t), \tag{1.5}$$

which is the pertinent approximate equation of the system. Here $g(t)$ is given by

$$g(t) = \beta(t) + \int_0^t f(t, \tau)\, \beta(\tau)\, d\tau, \tag{1.6}$$

where $\beta = \alpha$ when we assume the condition of initial straightness of the bar, i.e., when $w(t) \equiv 0$ at $t = 0^-$, or alternatively $\beta = \alpha + \eta w_0$ when an initial deviation w_0 is considered. In general, we will perform the analysis of the equations regardless the precise nature of the function $g(t)$.

This simple model contains all of the most interesting features of considerably more complex systems. In fact, in several papers by the author [1]-[7], it was shown that the investigation of time behavior and long-term stability of a large variety of viscoelastic structures—shells, plates, beam-columns, etc—can be reduced to consideration of an equation similar to (1.5) if the proper significance of the quantities appearing in that equation, is given. See for instance [4], [5], and [7]. Moreover, in subsequent work [8], [9], it was also shown what a fundamental role this equation plays in the discussion of long-term behavior and stability of more general, nonlinear, problems in the theory of hereditary phenomena.

2. Formulation of the Problem

The function η in Eq. (1.5) is assumed to range over the interval

$$0 < \eta < 1. \tag{2.1}$$

Clearly, the upper bound requirement $\eta < 1$ necessarily appears if we wish to avoid the trivial case of instantaneous instability originated as a result of the singularity of w at $\eta = 1$, see [9].

The general problem associated with Eq. (1.5) is to obtain conditions on the external forces and the function $f(t, \tau)$ such as to insure boundedness of w as $t \to \infty$. Of importance, next, is to obtain bounds and approximate representations for $w(t)$. The first goal clearly belongs to the class of Tauberian problems, and in fact, in previous work some Tauberian results have been used to investigate—under very general conditions—the behavior of w when η is a constant or exhibits an almost constant behavior. This was done in [1], [3], [9].

However, more interesting problems arise when the quantities are less restricted, i.e., some knowledge on their bounds or statistical behavior is available. In the last fashion, the case of η possessing a small randomly fluctuating component was investigated in [5] in connection with the stability of a viscoelastic rectangular plate. In this paper we generalize the results previously reported, in several directions.

Instead of Eq. (1.5) we shall consider the following, intimately related, integral equations

$$u(t) - \int_0^t \varphi(\tau) f(t, \tau)\, u(\tau)\, d\tau = g(t) \tag{2.2}$$

and

$$v(t) - \varphi(t) \int_0^t f(t, \tau)\, v(\tau)\, d\tau = G(t), \tag{2.3}$$

which under appropriate change of variables, namely

$$G(t) = g(t)\,\varphi(t), \tag{2.4}$$

$$\varphi(t) = \frac{\eta(t)}{1 - \eta(t)}, \tag{2.5}$$

$$u = (1 - \eta)\, w, \tag{2.6}$$

$$v = \eta w, \tag{2.7}$$

reduce to the original equation (1.5). Clearly

$$u + v = w \tag{2.8}$$

and

$$\frac{v}{u} = \varphi. \tag{2.9}$$

3. Almost Constant Behavior

We recall here some basic properties and asymptotic behavior of the solution of Eq. (2.2) which will be necessary in our subsequent analysis.

First we consider the case $\varphi = \varphi_0$ constant and f a function of the type $f(t - \tau)$. We can now enunciate the well-known theorem of Paley and Wiener [10] on the Volterra integral equation.

THEOREM 1 (Paley and Wiener). *If the limit*

$$c = \lim_{t\to\infty} g(t) \tag{3.1}$$

exists and f is absolutely integrable, i.e.,

$$\int_0^\infty | f(\tau) |\, d\tau < \infty \tag{3.2}$$

exists, then the limit of $u(t)$ given by Eq. (2.2) is

$$\lim_{t\to\infty} u(t) = \frac{c}{1 - \varphi_0 \int_0^\infty f(\tau)\, d\tau}, \tag{3.3}$$

if, and only if,

$$\varphi_0 \int_0^\infty e^{-st} f(t)\, dt \neq 1, \qquad \operatorname{Re} s > 0. \tag{3.4}$$

If we require the kernel to be nonnegative, then the condition expressed by Eq. (3.4) is replaced by

$$\varphi_0 \int_0^\infty f(t)\, dt < 1, \tag{3.5}$$

a condition which will play a fundamental role in our subsequent analysis.

If $g(t)$ does not have a limit but possess an upper bound, we may still use the following results. Introducing the operator

$$L_{\varphi_0}(\cdot) = (\cdot) - \varphi_0 \int_0^t f(t-\tau)\,(\cdot)\, d\tau, \tag{3.6}$$

Eq. (2.2) can be written

$$L_{\varphi_0} u = g(t). \tag{3.7}$$

It is not difficult to prove now that the inverse operator $L_{\varphi_0}^{-1}$ enjoys a monotone property in the sense that

$$g_1 \geqslant g_2 \geqslant 0 \Rightarrow L_{\varphi_0}^{-1} g_1 \geqslant L_{\varphi_0}^{-1} g_2 \geqslant 0.$$

Hence, if c is an upper bound of $g(t)$, $t \in R$, $R = [0, \infty)$,

$$u(t) \leqslant \frac{c}{1 - \varphi_0 \int_0^\infty f(t)\, dt}, \tag{3.8}$$

provided f is nonnegative and Eq. (3.5) is satisfied. Further, the solution of the Eq. (3.7) can be obtained in terms of the solution of the equation

$$L_{\varphi_0} x = H(t), \tag{3.9}$$

where $H(t)$ is the unit step function, by means of the relation

$$u(t) = g(t)\, x(0) + \varphi_0 \int_0^t g(t-\tau)\, dx(\tau), \tag{3.10}$$

under conditions for which the integral exists, see Ref. [11].

The forgoing results can be summarized in the following:

THEOREM 2. *Let c be an upper bound of the nonnegative function $g(t)$, $t \in R$, $f(t) > 0$ satisfies Eq.* (3.5), *$x(t)$ is the solution of Eq.* (3.9). *Then u in Eq.* (3.7) *is bounded in R and satisfies the Eqs.* (3.8) *and* (3.10).

When the kernel $f(t, \tau) = \varphi(\tau) f(t, \tau)$ is of a more general nature, the problem may be still tractable by means of a deep Tauberian result due to Pitt [12]. We restrict the kernel to belong to the class $\mathscr{C}$ of functions which may be approximated to functions of the type $k(t - \tau)$ in the sense that $F(t, \tau) \in \mathscr{C}(\sigma_1 \leqslant \sigma \leqslant \sigma_2, k(t))$ implies $F(t, \tau) e^{-\sigma(t-\tau)}$ and $k(t) e^{-\sigma t}$ belong to L and

$$\gamma(t) = \sup_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \int_0^\infty | F(t, \tau) - k(t - \tau) | e^{-\sigma(t-\tau)} d\tau, \tag{3.11}$$

is bounded and

$$\lim_{t \to \infty} \gamma(t) = 0. \tag{3.12}$$

Clearly, if $f(t, \tau) \in \mathscr{C}(\sigma_1 \leqslant \sigma \leqslant \sigma_2, k(t))$ and $\varphi(t) > 0$ has a limit φ_∞ as $t \to \infty$, then $\varphi(t) f(t, \tau)$ and $\varphi(\tau) f(t, \tau)$ belong to $\mathscr{C}(\sigma_1 \leqslant \sigma \leqslant \sigma_2, \varphi_\infty k(t))$. Then, application of the theorem by Pitt to Eq. (2.2) yields the following:

THEOREM 3. *Suppose $f(t, \tau) \in \mathscr{C}(0 \leqslant \sigma_1 \leqslant \sigma \leqslant \sigma_2, k(t))$, $\varphi(t) > 0$ has a finite limit φ_∞, $g(t)$ is bounded in R and*

$$\varphi_\infty \int_0^\infty e^{-st} k(t)\, dt \neq 1, \qquad \operatorname{Re} s > 0. \tag{3.13}$$

Then $u(t)$ in Eq. (2.2) is bounded and there exists Γ and G depending on $\varphi(\tau) f(t, \tau)$ and $g(t)$ such that

$$u \leqslant \Gamma G, \qquad t \in R. \tag{3.14}$$

We note that a similar result is obtained by considering the Eq. (2.3).

When the kernel is specified to be nonnegative, we further require the existence of nonnegative approximating functions $k(t)$ such that the condition given by Eq. (3.13) is replaced by

$$\varphi_\infty \int_0^\infty k(t)\, dt < 1. \tag{3.15}$$

We note that Theorem 3 embodies Theorems 1 and 2 in generality, but it is less precise as far as an estimate for an upper bound is concerned. For a discussion of upper estimates by assuming a monotone behavior of the kernel functions, see [3] and [4].

4. BOUNDS—MONOTONICITY OF THE INVERSE OPERATOR

When $\varphi(t)$ does not have a limit as $t \to \infty$, it is more difficult to predict the behavior of $u(t)$. In this section we discuss some lower and upper bounds for u based on the assumption that φ is a bounded function ranging over the interval

$$\varphi_1 \leqslant \varphi(t) \leqslant \varphi_2, \qquad \varphi_1 > 0, \qquad t \in R, \qquad \varphi_1, \varphi_2 \text{ constants}, \tag{4.1}$$

and that f is a nonnegative function of the type $f(t-\tau)$. Under these conditions we can prove the following elementary theorems.

THEOREM 4. *Let c be an upper bound of the nonnegative function $g(t)$, $t \in R$, and let φ satisfy Eq. (4.1). Then, if $f > 0$ and*

$$\varphi_2 \int_0^t f(t)\, dt < 1, \tag{4.2}$$

$u(t)$ in the equation

$$u(t) - \int_0^t \varphi(\tau) f(t-\tau)\, d\tau = g(t), \tag{4.3}$$

is bounded in R.

PROOF. Consider the sequence $\{u_n\}$ given by

$$u_0 = g$$

$$u_{n+1} = g + \int_0^t \varphi(\tau) f(t-\tau)\, u_n(\tau)\, d\tau. \tag{4.4}$$

By recalling the conditions of the hypotheses and observing that

$$\int_0^t f(\tau)\, d\tau \leqslant \int_0^\infty f(t)\, dt, \qquad t \in R, \tag{4.5}$$

it is easily proved in the familiar way that the series $\sum_{n=0}^{\infty} (u_{n+1} - u_n)$ converges absolutely and that the sequence $\{u_n\}$ converges to the bounded function u with an upper bound

$$u \leqslant \frac{c}{1 - \varphi_2 \int_0^\infty f(t)\, dt}, \qquad t \in R. \tag{4.6}$$

We may obtain a more precise upper bound, but we first prove the following

THEOREM 5. *Let $z(t)$, $t \in R$ be a continuous bounded function, $\varphi(t)$ and $f(t)$ satisfy the conditions of Theorem 4. Then*

$$z(t) \geqslant \int_0^t \varphi(\tau) f(t-\tau)\, z(\tau)\, d\tau, \qquad t \in R \tag{4.7}$$

implies $z(t) \geqslant 0$ in R.

Proof. From Eq. (4.7) we may have $z(0) \geqslant 0$. Suppose $z(0) = 0$. We first prove that there exists a finite interval $\rho = [0, t_1]$ such that $z(t) \geqslant 0$, $t \in \rho$. Taking into account Eqs. (4.1), (4.2), and (4.7) we see that $z(t) < 0$ in ρ implies

$$| z(t) | \leqslant \int_0^t \varphi(\tau) f(t-\tau) \, | z(\tau) | \, d\tau \leqslant | z(t_2) | \int_0^t \varphi(\tau) f(t-\tau) \, d\tau < | z(t_2) | ,$$

$$0 \leqslant t_2 \leqslant t, \qquad t \in \rho, \tag{4.8}$$

where

$$| z(t_2) | = \sup_{0 \leqslant \xi \leqslant t} | z(\xi) | , \qquad t \in \rho. \tag{4.9}$$

In the same way we may prove that

$$| z(t_2) | \leqslant | z(t_3) | , \qquad 0 \leqslant t_3 \leqslant t_2 , \tag{4.10}$$

which contradicts the equation (4.9). Then $z(t) \geqslant 0$, $t \in \rho$. The proof is completed by observing that $z(t)$ cannot change the sign. In fact, if there exists t_1 such that $z(t_1) = 0$, then from Eq. (4.7) we obtain

$$0 \geqslant \int_0^{t_1} \varphi(\tau) f(t_1 - \tau) \, z(\tau) \, d\tau, \tag{4.11}$$

which is a contradiction since φ and f are nonnegative functions, unless $z \equiv 0$, $t \in R$.

If $z(0) > 0$, we omit the first part of the proof.

We can now prove the following

Theorem 6. *Let $u(t)$ satisfy Eq. (4.3) and $f(t-\tau)$, $\varphi(t)$, $g(t)$, satisfy the conditions of Theorem 4. If x and y are the solutions of the equations*

$$x(t) - \varphi_1 \int_0^t f(t-\tau) \, x(\tau) \, d\tau = g(t) \tag{4.12}$$

and

$$y(t) - \varphi_2 \int_0^t f(t-\tau) \, y(\tau) \, d\tau = g(t), \tag{4.13}$$

respectively, then

$$0 \leqslant x(t) \leqslant u(t) \leqslant y(t), \qquad t \in R. \tag{4.14}$$

PROOF. Using Theorems 4 and 5 we can establish that

$$0 \leqslant y \leqslant \frac{c}{1 - \varphi_2 \int_0^\infty f(t)\, dt}. \tag{4.15}$$

Then, to prove that $u(t) \leqslant y(t)$, it is enough to prove that

$$z(t) = y(t) - u(t) \tag{4.16}$$

is positive in R. In fact, comparison of Eqs. (4.3), (4.13), and (4.16) yields the following equation for $z(t)$,

$$z(t) - \int_0^t \varphi(\tau) f(t-\tau)\, z(\tau)\, d\tau = \int_0^t (\varphi_2 - \varphi) f(t-\tau)\, v(\tau)\, d\tau. \tag{4.17}$$

The right hand member of Eq. (4.17) is positive and bounded, then z is bounded recalling Theorem 4 and $z(t) \geqslant 0$, $t \in R$, recalling Theorem 5. In the same way it may be proved that $\infty > u - x \geqslant 0$, completing the proof.

We observe that from the last Theorem we may establish the following:

LEMMA. *Let $\varphi(\tau)$, $f(t-\tau)$ satisfy the conditions of Theorem 4, and let c be a positive constant. If*

$$u \leqslant c + \int_0^t \varphi(\tau) f(t-\tau)\, u(\tau)\, d\tau, \tag{4.18}$$

then

$$0 \leqslant u \leqslant v \leqslant \frac{c}{1 - \varphi_2 \int_0^\infty f(t)\, dt}, \qquad t \in R, \tag{4.19}$$

where $v(t)$ is the solution of

$$v - \varphi_2 \int_0^t f(t-\tau)\, v(\tau)\, d\tau = c. \tag{4.20}$$

This is one of the ways that the fundamental lemma by R. Bellman [13] can be generalized. See also [14].

5. Time Averages

We first consider a simple example. Let

$$f(t, \tau) = a\delta e^{-\delta(t-\tau)}, \qquad a, \delta > 0. \tag{5.1}$$

be the memory function in Eq. (2.2). It is not difficult to prove that the solution u can be represented by

$$u(t) = g(t) + a\delta e^{-\delta t[1-a\sigma(t)]} \int_0^t \varphi(\tau) g(\tau) e^{\delta\tau[1-a\sigma(t)]} \, d\tau, \tag{5.2}$$

where

$$\sigma(t) = \frac{1}{t} \int_0^t \varphi(\tau) \, d\tau, \tag{5.3}$$

is the Cesaro mean of function $\varphi(t)$. If the limit

$$\phi = \lim_{t\to\infty} \sigma(t) \tag{5.4}$$

exists, we may set $\varphi = \phi[1 + \psi(t)]$ where ψ is a function with mean zero. Now Eq. (5.4) may be written

$$u(t) = g(t) + a\delta \exp\left[-\delta(1 - a\phi)\, t + a\phi\delta \int_0^t \psi \, d\xi\right]$$
$$\times \int_0^t \varphi(\tau) g(\tau) \exp\left[\delta(1 - a\phi)\,\tau - a\phi\delta \int_0^t \psi \, d\xi\right] d\tau. \tag{5.5}$$

If for instance ψ is a periodic function taking the values $\pm \nu$ at equal intervals of time, and g is an almost constant function with $\lim_{t\to\infty} g = 1$, then, for large t function u will oscillate from the value

$$u_1 = 1 + \frac{a\phi(1 + \nu)}{1 - a\phi(1 + \nu)},$$

to the the value

$$u_2 = 1 + \frac{a\phi(1 - \nu)}{1 - a\phi(1 - \nu)},$$

at equal intervals of time. So the mean $\bar{u}$ for large t is

$$\bar{u} = \frac{\frac{1}{2}}{1 - a\phi(1 + \nu)} + \frac{\frac{1}{2}}{1 - a\phi(1 - \nu)}. \tag{5.6}$$

This is a particular case of a more general result. In fact, consider the equation

$$u - \phi \int_0^t [1 + \psi(\tau)] f(t - \tau) \, u(\tau) \, d\tau = g(t),$$

where ϕ is a constant and ψ takes the value $\nu > 0$ with probability p_1 and the value $-\nu$ with probability $1 - p_1$. Suppose $\lim_{t\to\infty} g = 1$, $f > 0$ and

$$\phi(1 + \nu) \int_0^\infty f(t)\, dt < 1. \tag{5.7}$$

The mean value of u is then given by

$$\bar{u} \sim \frac{p_1}{1 - \phi(1 + \nu)} + \frac{(1 - p_1)}{1 - \phi(1 - \nu)}, \qquad \text{for large } t, \tag{5.8}$$

as it may be easily established from consideration of Theorem 6.

6. The Stochastic Case—General Remarks

We are interested in the representation of the mean values of u and v given by the Eqs. (2.2) and (2.3) under more general conditions than those considered in the last section. To accomplish this purpose two different techniques will be used. An iterative procedure in the fashion of the method of successive approximations will be applied on Eq. (2.3), while a technique of truncated hierarchies in the way it was suggested by J. Richardson [15] to treat stochastic differential equations, will be used on Eq. (2.2).

We consider φ a function of the random variable η, not necessarily defined as in Eq. (2.5). We may consider different cases for the function $g(t)$.

(1) $g(t)$ not random.

(2) $g(\alpha)$, depending on a random function α statistically independent of η.

(3) $g(\alpha, \eta) = g_1(\alpha) + g_2(\eta)$ depending on both random functions α and η.

Only the first case will be considered, but the procedures we use are applicable to the general case without modification. Function φ will be considered a stationary random process and the symbol $\langle\cdot\rangle$ will be used to denote ensemble averages. It is assumed that $\langle\varphi\rangle$ exists and

$$\varphi = \phi[1 + \psi], \tag{6.1}$$

where $\phi = \langle\varphi\rangle$ and ψ is a purely random function with mean zero. It is convenient to introduce the not random operator

$$L_\phi(\cdot) = (\cdot) - \phi \int_0^t f(t - \tau)\,(\cdot)\, d\tau \tag{6.2}$$

and the inverse operator

$$L_\phi^{-1}(\cdot) = (\cdot) + \int_0^t h(t - \tau)\,(\cdot)\, d\tau \tag{6.3}$$

such that $L_\phi^{-1}y = x \Rightarrow y = L_\phi x$ and such that y bounded implies x bounded, in R. This is insured provided f is absolutely integrable and

$$\phi \int_0^\infty f(t)\, e^{-st}\, dt \neq 1, \qquad \operatorname{Re} s > 0. \tag{6.4}$$

We consider here only nonnegative kernels so that the last condition is replaced by

$$\phi \int_0^\infty f(t)\, dt < 1. \tag{6.5}$$

This equation insures that

$$\int_0^\infty h(t)\, dt < \infty \tag{6.6}$$

exists.

7. Successive Approximations[1]

Let $v(t)$ be the solution of the equation

$$v(t) - \varphi(t) \int_0^t f(t-\tau)\, v(\tau)\, d\tau = g(t)\, \varphi(t). \tag{7.1}$$

We consider the sequence $\{v_n\}$ defined by

$$\begin{aligned} L_\phi v_0 &= g\phi \\ L_\phi v_{n+1} &= g\varphi(t) + \phi\psi(t) \int_0^t f(t-\tau)\, v_n(\tau)\, d\tau, \qquad n = 0, 1, \ldots . \end{aligned} \tag{7.2}$$

If we introduce now the functions x_n defined by

$$L_\phi x_0 = g\phi \tag{7.3.1}$$

$$L_\phi x_1 = g\phi\psi + \phi\psi \int_0^t f(t-\tau)\, x_0(\tau)\, d\tau = \psi x_0 \tag{7.3.2}$$

$$L_\phi x_2 = \phi\psi \int_0^t f(t-\tau)\, x_1(\tau)\, d\tau = \psi x_1 - \psi^2 x_0 \tag{7.3.3}$$

$$L_\phi x_3 = \phi\psi \int_0^t f(t-\tau)\, x_2(\tau)\, d\tau = \psi x_2 - \psi^2 x_1 + \psi^3 x_0 , \tag{7.3.4}$$

$$\vdots$$

[1] It should be noted that the technique of successive approximations as it is used in this section is completely equivalent to expansion in a small parameter multiplying the fluctuation terms. The author wishes to thank Dr. John Richardson for the comment on this equivalence.

it may be proved by substitution that the nth order approximation v_n is given by

$$v_n = x_0 + x_1 + \cdots + x_n , \qquad n = 0, 1, \dots . \tag{7.4}$$

Invertion of Eq. (7.3.2) yields

$$x_1 = \psi(t)\, x_0(t) + \int_0^t h(t-\tau)\, \psi(\tau)\, x_0(\tau)\, d\tau, \tag{7.5}$$

which substituted in Eq. (7.3.3) furnishes

$$L_\phi x_2 = \psi(t) \int_0^t h(t-\tau)\, \psi(\tau)\, x_0(\tau)\, d\tau. \tag{7.6}$$

Recalling Eqs. (7.3.1), (7.3.2), (7.4), and (7.6) we obtain

$$L_\phi v_2 = \phi g(t) + \psi(t)\, v_0(t) + \psi(t) \int_0^t h(t-\tau)\, \psi(\tau)\, v_0(\tau)\, d\tau, \tag{7.7}$$

where v_0 is a not random function which satisfies the integral equation

$$L_\phi v_0 = g\phi.$$

Taking averages in Eq. (7.7)

$$L_\phi \langle v_2 \rangle = \phi g(t) + \int_0^t C(t-\tau)\, h(t-\tau)\, v_0(\tau)\, d\tau, \tag{7.8}$$

where $C(\tau) = \langle \psi(t)\, \psi(t+\tau) \rangle$ is the autocorrelation function, we obtain an integral equation for the second-order approximation of the mean value of v. The same procedure can be used to obtain the corresponding integral equation for the higher order approximations of the mean, for instance,

$$\begin{aligned} L_\phi \langle v_3 \rangle &= L_\phi \langle v_2 \rangle - \langle \psi^3 \rangle\, x_0 \\ &\quad + \int_0^t h(t-\tau)\, d\tau \int_0^\tau \langle \psi(t)\, \psi(\tau)\, \psi(\xi) \rangle\, h(\tau - \xi)\, x_0(\xi)\, d\xi, \end{aligned} \tag{7.9}$$

etc.

8. Convergence and Stability of the Approximations

Considering Eqs. (7.3) and taking into account the results obtained in Section 3, we may easily prove that

$$\| x_0 \| \leqslant \| g \| \phi \Gamma$$

$$\| x_n \| \leqslant \| g \| \phi \Gamma \frac{1}{A} \| \psi \|^n \Gamma^n A^n, \tag{8.1}$$

where

$$\| \cdot \| = \sup_{t \in R} | \cdot | , \tag{8.2}$$

$$A = \phi \int_0^\infty f(t)\, dt, \tag{8.3}$$

$$\Gamma^{-1} = 1 - A. \tag{8.4}$$

Then the series

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} (v_{n+1} - v_n), \tag{8.5}$$

converges absolutely and the sequence $\{v_n\}$ converges to the function v, $t \in R$, with an upper bound

$$\| v \| \leqslant \frac{\| g \| \, \| \varphi \|}{1 - \| \varphi \| \int_0^\infty f(t)\, dt}, \qquad 8.(6)$$

provided

$$\| \varphi \| \int_0^\infty f(t)\, dt < 1. \tag{8.7}$$

We are now interested in the accuracy of the approximations obtained with the foregoing procedure. It might be expected that a method of successive approximations yields good results by truncation at low orders, provided the randomly fluctuating component of the process is small enough, i.e., $\| \psi \| \ll 1$. We wish to show that this is not true, i.e., that smallness of $\| \psi \|$ alone is not sufficient to insure accuracy in the present case. In fact, consider the simple process studied in Section 5, and assume that ψ takes the values $\pm \nu$ with equal probability. Then, if we assume $\phi \lim_{t \to \infty} g = 1$, the asymptotic value of the mean $\langle v \rangle$, where v satisfies Eq. (7.1), is shown to be

$$\langle v \rangle \sim \frac{1}{2} \left[\frac{1 + \nu}{1 - \phi(1 + \nu) \int_0^\infty f\, d\tau} + \frac{1 - \nu}{1 - \phi(1 - \nu) \int_0^\infty f\, d\tau} \right], \quad \text{for large } t, \tag{8.8}$$

while the asymptotic value of the second-order approximation $\langle v_2 \rangle$ given by the Eq. (7.8)—under the same conditions—is found to be

$$\langle v_2 \rangle \sim \frac{1}{1 - \phi \int_0^\infty f\, d\tau} + \frac{\nu^2 \phi \int_0^\infty f(t)\, dt}{\left(1 - \phi \int_0^\infty f\, d\tau\right)^2}, \quad \text{for large } t. \tag{8.9}$$

It is recognized that when $\phi(1+\nu)\int_0^\infty f\,dt = 1$, the exact value given by Eq. (8.8) becomes unbounded while the approximate result still remains finite. So, in general we must require not only $\|\psi\| \ll 1$ but also

$$\|\varphi\| \int_0^\infty f(t)\,dt \ll 1,$$

in order to obtain accuracy. The same remark applies also for higher order approximations since they satisfy integral equations of the type

$$L_\phi\langle v_n\rangle = F(\langle v_{n-1}\rangle, \langle v_{n-2}\rangle, \ldots, v_0),$$

which leads to asymptotic bounded results when $\phi \int_0^\infty f(t)\,dt < 1$.

9. Truncated Hierarchies

We consider an alternative technique to obtain approximate solutions of random integral equations of the type (2.2) or (2.3). For further discussion of this method, see Ref. [15]. Consider as an example, Eq. (2.2).

Under the general remarks of Section 6, we consider the unaveraged hierarchy

$$\begin{aligned}
L_\phi u &= g + \phi \int_0^t \psi(\tau) f(t-\tau)\, u(\tau)\, d\tau,\\
L_\phi(\psi_1 u) &= \psi_1 g + \phi \int_0^t \psi_1\psi(\tau) f(t-\tau)\, u(\tau)\, d\tau,\\
L_\phi(\psi_1\psi_2 u) &= \psi_1\psi_2 g + \phi \int_0^t \psi_1\psi_2\psi(\tau) f(t-\tau)\, u(\tau)\, d\tau, \qquad (9.1)\\
&\;\vdots
\end{aligned}$$

obtained after multiplication of the original equation by $\psi_1\psi_2 \cdots \psi_n$, where $\psi_i = \psi(t_i)$, $i = 1, 2, \ldots$. Taking averages in Eq. (9.1) we obtain the averaged hierarchy

$$\begin{aligned}
L_\phi\langle u\rangle &= g + \phi \int_0^t f(t-\tau)\,\langle \psi(\tau)\, u(\tau)\rangle\, d\tau\\
L_\phi\langle \psi_1 u\rangle &= \phi \int_0^t f(t-\tau)\,\langle \psi_1\psi(\tau)\, u(\tau)\rangle\, d\tau\\
L_\phi\langle \psi_1\psi_2 u\rangle &= g\langle\psi_1\psi_2\rangle + \phi \int_0^t f(t-\tau)\,\langle \psi_1\psi_2\psi(\tau)\, u(\tau)\rangle\, d\tau. \qquad (9.2)\\
&\;\vdots
\end{aligned}$$

Assuming local independence of $\psi_1\psi_2 \cdots \psi_n\psi(t)$ and $u(t)$, i.e.,

$$\langle\psi_1\psi_2 \cdots \psi_n\psi(t)\, u(t)\rangle \sim \langle\psi_1\psi_2 \cdots \psi_n\psi(t)\rangle \langle u(t)\rangle, \tag{9.3}$$

the first meaningful approximation is obtained by truncation at the second order. In so doing we obtain the following consistent set of integral equations

$$L_\phi\langle u\rangle = g + \phi \int_0^t f(t-\tau) \langle\psi(\tau)\, u(\tau)\rangle\, d\tau$$

$$L_\phi\langle\psi_1 u\rangle = \phi \int_0^t f(t-\tau)\, C(t_1-\tau) \langle u(\tau)\rangle\, d\tau. \tag{9.4}$$

In order to eliminate the term $\langle\psi u\rangle$ we observe that the first equation (9.4) can be written

$$L_\phi\langle u\rangle = g + \langle\psi u\rangle - L_\phi - \langle\psi u\rangle, \tag{9.5}$$

and that

$$\langle\psi u\rangle = \phi \int_0^t f(t-\tau)\, C(t-\tau) \langle u(\tau)\rangle\, d\tau$$
$$+ \phi \int_0^t h(t-\tau)\, d\tau \int_0^\tau f(\tau-\xi)\, C(\tau-\xi) \langle u(\xi)\rangle\, d\xi. \tag{9.6}$$

Consequently, we obtain

$$L_\phi\langle u\rangle - \phi \int_0^t h(t-\tau)\, d\tau \int_0^\tau C(\tau-\xi) f(\tau-\xi) \langle u(\xi)\rangle\, d\xi = g(t), \tag{9.7}$$

which is the pertinent approximate integral equation for the mean $\langle u\rangle$. A discussion of the solution of Eq. (9.7) is beyond the scope of this paper. We wish only to show, in a limiting case, the nature of the approximation involved. In fact, we consider the autocorrelation function to be a constant, i.e., $C = \nu^2$. Since $-\phi f(t)$ and $h(t)$ are reciprocal kernels, the following relation

$$\phi \int_0^t h(t-\tau)\, d\tau \int_0^\tau f(\tau-\xi)\, x(\xi)\, d\xi$$
$$= \int_0^t h(t-\tau)\, x(\tau)\, d\tau - \phi \int_0^t f(t-\tau)\, x(\tau)\, d\tau \tag{9.8}$$

holds for all x. Then, if we assume $C = \nu^2$, Eq. (9.7) reduces to

$$\langle u\rangle - \int_0^t [\phi(1-\nu^2) f(t-\tau) + \nu^2 h(t-\tau)] \langle u(\tau)\rangle\, d\tau = g(t). \tag{9.9}$$

Suppose g has a limit as $t \to \infty$ and more precisely, $\lim_{t\to\infty} g = 1$. Then we have

$$\langle u \rangle \sim \frac{1}{1 - \phi(1 - \nu^2) \int_0^\infty f(t)\, dt - \nu^2 \int_0^\infty h(t)\, dt}, \quad \text{for large } t, \tag{9.10}$$

if, and only if,

$$\phi(1 - \nu^2) \int_0^\infty f(t)\, dt + \nu^2 \int_0^\infty h(t)\, dt < 1. \tag{9.11}$$

Since

$$1 + \int_0^\infty h(t)\, dt = \frac{1}{1 - \phi \int_0^\infty f(t)\, dt}, \tag{9.12}$$

Eq. (9.10) reduces to

$$\langle u \rangle \sim \frac{\frac{1}{2}}{1 - \phi(1 + \nu) \int_0^\infty f(t)\, dt} + \frac{\frac{1}{2}}{1 - \phi(1 - \nu) \int_0^\infty f(t)\, dt}, \quad \text{for large } t, \tag{9.13}$$

which is the exact asymptotic value of the mean $\langle u \rangle$ when ψ takes the values $\pm\, \nu$ with equal probability.

The condition of stability given by Eq. (9.11) reduces to

$$\frac{1}{\dfrac{1}{1 - \phi(1 + \nu) \int_0^\infty f(t)\, dt} + \dfrac{1}{1 - \phi(1 - \nu) \int_0^\infty f(t)\, dt}} > 0, \tag{9.14}$$

which is satisfied provided

$$\phi(1 + \nu) \int_0^\infty f(t)\, dt < 1. \tag{9.15}$$

We note that the same asymptotic result is obtained if the autocorrelation function tends asymptotically to a constant value.

10. Discussion

Consideration of long term stability of some linear hereditary phenomena has motivated the investigation of a Volterra integral equation of the type

$$L_\varphi(u) = u - \int_0^t \varphi(\tau) f(t - \tau)\, u(\tau)\, d\tau = g(t).$$

It is shown that the inverse operator L_φ^{-1} enjoys some monotone properties in the sense that $g_1 \geqslant g_2 \Rightarrow L_\varphi^{-1} g_1 \geqslant L_\varphi^{-1} g_2$ and that $\varphi_1 \geqslant \varphi_2 \Rightarrow L_{\varphi_1}^{-1} g \geqslant L_{\varphi_2}^{-1} g$. Under the conditions which insure monotonicity, bounds for $u(t)$, $t \in R$, have been obtained in Section 4.

The stochastic version of this problem have been tackled by two alternative methods. A technique of successive approximations yields a sequence of not random integral equations of the type

$$L_\phi \langle u_n \rangle = F(\langle u_{n-1} \rangle, \langle u_{n-2} \rangle \cdots u_0), \qquad n = 1, 2, \ldots,$$

for the successive approximations of the mean $\langle u_n \rangle$. Conditions for the convergence of the procedure have been investigated, and the accuracy of the lower order approximations has been discussed. It was shown that the method yields good results provided

$$\| \varphi \| \int_0^\infty f(t)\, dt \ll 1, \qquad \text{and} \qquad \| \psi \| \ll 1.$$

A technique of truncated hierarchy surmounts this restriction although yields to a more involved integral equation for the approximated mean $\langle u \rangle$. It was shown by consideration of a simple stochastic process that this technique yields to the exact value of the mean for large values of t.

References

1. J. N. Distefano. Sulla stabilità in regime viscoelastico a comportamento lineare. *Rend. Acc. Naz. Lincei, Ser. VIII,* **27** (1959), 205–211 and 356–361.
2. J. N. Distefano. Creep deflections in concrete and reinforced concrete columns. Publications. Int. Assoc. for Bridges and Str. Eng. Vol. 21, pp. 37–48, 1961.
3. J. N. Distefano. Ancora sulla stabilità asintotica delle deflezioni di una trave viscoelastica. *Rend. Acc. Naz. Lincei, Ser. VIII,* **35** (1963), 504–508.
4. J. N. Distefano. Creep Buckling of Slender Columns. *Proc. Amer. Soc. Civil Eng., J. Str. Div.* **91** (1965), 127–150.
5. J. N. Distefano. Creep buckling of viscoelastic structures under stochastic loads. Pre-print of the Int. Symp. RILEM on Repeated Loading of Materials and Str., Mexico City, 1966.
6. J. N. Distefano and M. H. Gradowczyk. Creep behavior of homogeneous anisotropic prismatic shells. Proc. Symp. Non-Classical Shell Problems, pp. 409–427, IASS, 1963.
7. M. H. Gradowczyk and J. N. Distefano. The viscoelastic theory of thin shallow shells. *Ing. Arch.* **34** (1965), 304–312.
8. J. N. Distefano and J. Sackman. On asymptotic stability of nonlinear hereditary phenomena. *Q.A.M.* **24** (1966), 2.
9. J. N. Distefano and J. Sackman. Stability analysis of a nonlinear viscoelastic column. *Z.A.M.M.* **47** (1967) *Heft* **6**, pp. 349–358.

10. R. E. PALEY AND N. WIENER. Fourier transforms in the complex domain. *Amer. Math. Soc. Colloq. Publ.* **XIX** (1934).
11. R. BELLMAN AND K. COOKE. "Differential-Difference Equations." Academic Press, New York, 1963.
12. H. R. PITT. "Tauberian Theorems." Oxford Univ. Press, 1958.
13. R. BELLMAN. "Stability Theory of Differential Equations." McGraw-Hill, New York, 1954.
14. E. F. BECKENBACH AND R. BELLMAN. "Inequalities." Springer, Berlin, 1961.
15. J. M. RICHARDSON. The application of truncate hierarchy techniques in the solution of a stochastic linear differential equation. *Amer. Math. Soc. Proc. Symp. Appl. Math.* **XVI** (1964).